SECTION 8.7: OTHER METHODS OF INTEGRATION

EXAMPLE 1: Consider the the integration formulas below:

- $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
- $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\sec^{-1}\left(\frac{|u|}{a}\right) + C$
- $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\left(\frac{u}{a}\right) + C$
- $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{1}{u}\sqrt{u^2-a^2} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

Use a substitution (or two!) and one or more of the formulas above to help you find the following integrals:

1. $\displaystyle\int \frac{\sqrt{9x^2-4}}{x^2}\,dx$

Ans: $\displaystyle -\frac{1}{x}\sqrt{9x^2-4} + 3\ln\left|3x+\sqrt{9x^2-4}\right| + C$

2. $\displaystyle\int \sqrt{1-e^{2x}}\,dx$

Ans: $\displaystyle \sqrt{1-e^{2x}} - \ln\left(\frac{1+\sqrt{1-e^{2x}}}{e^x}\right) + C = x + \sqrt{1-e^{2x}} - \ln\left(1+\sqrt{1-e^{2x}}\right) + C$

EXAMPLE 2: (VIDEO) Use the reduction formula:

$$\int \sin^n(u)\,du = -\frac{1}{n}\sin^{n-1}(u)\cos(u) + \frac{n-1}{n}\int \sin^{n-2}(u)\,du$$

to help you find $\displaystyle\int \sin^6(2x)\,dx$.

Ans: $\displaystyle -\frac{1}{12}\sin^5(2x)\cos(2x) - \frac{5}{48}\sin^3(2x)\cos(2x) - \frac{15}{96}\sin(2x)\cos(2x) + \frac{15}{48}x + C$

EXAMPLE 3: Recall the product-to-sum identities:

- $\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
- $\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$
- $\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Use an appropriate identity to show $\int_{-\pi}^{\pi} \sin(2x)\cos(5x)\,dx = 0$

EXAMPLE 4 (VIDEO): The substitution infamous '$z = \tan\left(\frac{x}{2}\right)$' substitution is as follows:

$$z = \tan\left(\frac{x}{2}\right), \qquad dx = \frac{2}{1+z^2}\,dz, \qquad \cos(x) = \frac{1-z^2}{1+z^2}, \qquad \sin(x) = \frac{2z}{1+z^2}.$$

This substitution transforms rational functions of trigonometric functions into rational functions of 'z'.

Use the $z = \tan\left(\frac{x}{2}\right)$ substitution to help you find: $\int \frac{1}{4 - 3\cos(x)}\,dx$.

Ans: $\frac{2}{\sqrt{7}}\tan^{-1}\left(\sqrt{7}\tan\left(\frac{x}{2}\right)\right) + C$

HYPERBOLIC SUBSTITUTIONS: Recall the following identities relating the hyperbolic functions:

- $\cosh^2(t) - \sinh^2(t) = 1$ so that $1 + \sinh^2(t) = \cosh^2(t)$ or, more generally:$a^2 + a^2\sinh^2(t) = a^2\cosh^2(t)$.
- $\cosh^2(t) - \sinh^2(t) = 1$ so that $\sinh^2(t) - 1 = \cosh^2(t)$ or, more generally: $a^2\sinh^2(t) - a^2 = a^2\cosh^2(t)$
- $1 - \tanh^2(t) = \operatorname{sech}^2(t)$ or, more generally: $a^2 - a^2\tanh^2(t) = a^2\operatorname{sech}^2(t)$.

These 'Pythagorean-like' identities mean that hyperbolic functions are useful in the same way their (circular) trigonometric functions cousins in simplyfing integrals involving two term radicands as we see in the next example.

EXAMPLE 5 (VIDEO): Consider the integral $\int \frac{1}{\sqrt{x^2+9}}\,dx$.

1. Solve this integral using the techniques of Section 8.4.

   Ans: $\ln\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right| + C = \ldots = \ln\left(x + \sqrt{x^2+9}\right) + C$

2. Solve this integral using the substitution $x = 3\sinh(t)$ and simplify using a hyperbolic identity.

   HINT: If $x = 3\sinh(t)$, then $\sinh(t) = \frac{x}{3}$ so that $t = \sinh^{-1}\left(\frac{x}{3}\right)$. Also recall: $D_t\left[\sinh(t)\right] = \cosh(t)\ \ldots$

3. Use a computer algebra system to determine I $\int \frac{1}{\sqrt{x^2+9}}\,dx$. Which answer, if any, does it match?

HOMEWORK: Section 8.3: 67 - 73 odd, Section 8.7: 7 - 39 every other odd.